Partial Differential Equations (WBMA008-05)

Representative exam problems

University of Groningen

Instructions

1. The use of calculators is *not* allowed. It is allowed to use a "cheat sheet" (one sheet A4, handwritten, wet ink, both sides).
2. All answers need to be accompanied with an explanation or a calculation: only answering "yes", "no", or "42" is not sufficient.

Problem 1

Consider the following nonuniform transport equation:

$$\frac{\partial u}{\partial t} + (1+x^2)\frac{\partial u}{\partial x} = 0, \quad u(0,x) = \cos(x).$$

(a) Compute *all* characteristic curves; express the answer in the form $x = x(t)$.

(b) Determine the region D of the (t,x)-plane in which the solution is determined by the initial condition.

(c) Compute the solution in the region D.

Problem 2

Compute the real Fourier coefficients a_k and b_k for 2-periodic extension of the following function:

$$f : [-1,1] \to \mathbb{R}, \quad f(x) = 1 - x^2.$$

Problem 3

Use the d'Alembert formula to solve the wave equation

$$u_{tt} = 4u_{xx}, \quad -\infty < x < \infty, \quad t > 0,$$

with initial conditions $u(0,x) = e^{-x^2}$ and $u_t(0,x) = xe^{-x^2}$.

Problem 4

Consider the following damped wave equation with $0 < a < c$:

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} - 2a\frac{\partial u}{\partial t}, \quad u(t,0) = u(t,\pi) = 0.$$

Determine all nontrivial solutions of the form $u(t,x) = w(t)v(x)$.

Problem 5

(a) Show that $u(x,y) = e^y \cos(x)$ is a harmonic function.

(b) Compute the maximum and minimum values of u on $[-\frac{1}{2}\pi, \frac{1}{2}\pi] \times [-1,1]$.

(c) Compute the integral $\int_{-\pi}^{\pi} e^{\sin t} \cos(\cos t)\, dt$.

Problem 6

Compute the Green's function for the following boundary value problem:

$$\frac{d}{dx}\left(\frac{1}{1+x^2}\frac{du}{dx}\right) = f(x), \quad u(0) = 0, \quad u(1) = 0.$$

Problem 7

Recall the following function:

$$G_0(x,y;\xi,\eta) = -\frac{1}{2\pi} \log \|(x,y) - (\xi,\eta)\|,$$

where $\|\cdot\|$ denotes the Euclidean norm. Use this function and the method of images to construct the Green's function for Poisson's equation on the domain $\Omega = \{(x,y) \in \mathbb{R}^2 : y > x\}$.

Problem 8

Consider the following equation for $-\infty < x < \infty$ and $t > 0$:

$$\frac{\partial^2 u}{\partial t^2} + 2\frac{\partial u}{\partial t} = -u, \quad u(0,x) = f(x), \quad \frac{\partial u}{\partial t}(0,x) = g(x).$$

Use Fourier transforms to solve this equation; express the final answer $u(t,x)$ in terms of the functions f and g.

End of test

Solution of problem 1

(a) The characteristic curves are found by solving the equation $dx/dt = 1 + x^2$. Separation of variables gives

$$\int \frac{1}{1+x^2}\,dx = \int dt \quad \text{and thus} \quad \arctan(x) = t + k,$$

where $k \in \mathbb{R}$ is an arbitrary constant. By inverting the arctan function we can express the characteristic curves as follows:

$$t \mapsto (t, \tan(t+k)).$$

(b) Along a characteristic curve the solution u is constant. To determine the value of this constant we need to use the initial condition and that is only possible when the characteristic curve intersects the x-axis.

Note that the characteristic curves intersect the x-axis if and only if $-\frac{1}{2}\pi < k < \frac{1}{2}\pi$. This means that the solution $u(t,x)$ is only determined by the initial condition in the region

$$D = \bigcup_{k\in(-\pi/2,\pi/2)} \big\{(t, \tan(t+k)) \,:\, t \in (-\pi/2, \pi/2)\big\}.$$

Alternatively, we can write this region in the following simpler form:

$$D = \big\{(t,x) \in \mathbb{R}^2 \,:\, -\tfrac{1}{2}\pi + \arctan(x) < t < \tfrac{1}{2}\pi + \arctan(x)\big\}.$$

(c) *Method 1.* In the region D the solution is given by

$$u(t,x) = \cos(\beta^{-1}(\beta(x) - t)) = \cos(\tan(\arctan(x) - t)).$$

Method 2. Assume that $(\bar{t},\bar{x}) \in D$. Observe that this point lies on the characteristic curve with $k = \arctan(\bar{x}) - \bar{t}$. This curve intersects the x-axis in the point $(0, \tan(\arctan(\bar{x}) - \bar{t}))$. Since solutions are constant along the characteristic curve we have

$$u(\bar{t},\bar{x}) = u(0, \tan(\arctan(\bar{x}) - \bar{t})) = \cos(\tan(\arctan(\bar{x}) - \bar{t})).$$

Dropping the bars gives the desired expression.

Solution of problem 2

The Fourier coefficients are given by

$$\begin{aligned}
a_0 &= \int_{-1}^{1} f(x)\,dx = \frac{4}{3}, \\
a_k &= \int_{-1}^{1} f(x)\cos(k\pi x)\,dx = (-1)^{k+1}\frac{4}{k^2\pi^2}, \\
b_k &= \int_{-1}^{1} f(x)\sin(k\pi x)\,dx = 0.
\end{aligned}$$

Solution of problem 3

The formula of d'Alembert with $c = 2$, $f(x) = e^{-x^2}$, and $g(x) = xe^{-x^2}$ gives

$$\begin{aligned} u(t,x) &= \frac{f(x-ct)+f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(z)\,dz \\ &= \frac{e^{-(x-2t)^2}+e^{-(x+2t)^2}}{2} + \frac{1}{4}\int_{x-2t}^{x+2t} ze^{-z^2}\,dz \\ &= \frac{e^{-(x-2t)^2}+e^{-(x+2t)^2}}{2} + \frac{1}{4}\Big[-\frac{1}{2}e^{-z^2}\Big]_{x-2t}^{x+2t} \\ &= \frac{e^{-(x-2t)^2}+e^{-(x+2t)^2}}{2} - \frac{e^{-(x+2t)^2}-e^{-(x-2t)^2}}{8} \\ &= \frac{5e^{-(x-2t)^2}+3e^{-(x+2t)^2}}{8}. \end{aligned}$$

Solution of problem 4

Substituting the ansatz $u(t,x) = w(t)v(x)$ in the equation gives

$$\frac{w''(t)+2aw'(t)}{c^2w(t)} = \frac{v''(x)}{v(x)}.$$

Since the variables t and x are independent this equality can only hold if both sides are constant. Hence, we obtain the equations

$$w''(t)+2aw'(t)-c^2\lambda w(t) = 0 \quad \text{and} \quad v''(x)-\lambda v(x) = 0.$$

In addition, we have the boundary conditions $v(0) = v(\pi) = 0$.

We now distinguish between three different cases.

- If $\lambda = \omega^2 > 0$, then $v(x) = a\cosh(\omega x)+b\sinh(\omega x)$ but the boundary conditions imply that $a = b = 0$. So in this case we only obtain trivial solutions.
- If $\lambda = 0$, then $v(x) = a+bx$ but the boundary conditions imply that $a = b = 0$. So in this case we only obtain trivial solutions.
- If $\lambda = -\omega^2 < 0$, then $v(x) = a\cos(\omega x)+b\sin(\omega x)$. The boundary conditions then imply that $a = 0$ and $b\sin(\omega\pi) = 0$. In order to obtain nontrivial solutions, we conclude that $\omega = k \in \mathbb{N}$.

For $\lambda = -k^2$ with $k \in \mathbb{N}$ we get the following equation:

$$w''(t)+2aw'(t)+k^2c^2\lambda w(t) = 0.$$

Trying a solution of the form $w(t) = e^{\mu t}$ gives $\mu^2+2a\mu+k^2c^2 = 0$. Since we have assumed that $0 < a < c$ we get the solutions

$$\mu = -a \pm i\sqrt{k^2c^2-a^2}.$$

Finally, we conclude that the nontrivial solutions of the damped wave equation are given by

$$\begin{aligned} u_k(t,x) &= e^{-at}\cos\big(\sqrt{k^2c^2-a^2}t\big)\sin(kx) \\ u_k(t,x) &= e^{-at}\sin\big(\sqrt{k^2c^2-a^2}t\big)\sin(kx), \end{aligned}$$

where $k \in \mathbb{N}$.

Solution of problem 5

(a) A straightforward computation shows that $u_{xx}+u_{yy}=0$, which means that u is harmonic.

(b) Since u is harmonic on the domain $\Omega=[-\frac{1}{2}\pi,\frac{1}{2}\pi]\times[-1,1]$ the maximum and minimum value is attained on the boundary of Ω; these values are e and 0, respectively.

(c) For a harmonic function we have the mean value property:

$$\frac{1}{2\pi r}\int_{-\pi}^{\pi} u(a+r\cos(t),b+r\sin(t))r\,dt = u(a,b).$$

In words: the average value of u computed over a circle of radius r equals the value of u at the midpoint of that circle. This immediately implies that

$$\int_{-\pi}^{\pi} e^{\sin t}\cos(\cos t)\,dt = 2\pi\,u(0,0) = 2\pi.$$

Solution of problem 6

Setting $p(x)=1/(1+x^2)$ we can write the differential equation as $L[u]=f$ where

$$L[u]=(pu')'=pu''+p'u'.$$

The general solution of the homogeneous differential equation $L[u]=0$ is

$$u(x)=a+bx+\frac{b}{3}x^3,$$

where $a,b\in\mathbb{R}$ are arbitrary constants. Note that the solution $u_1(x)=3x+x^3$ satisfies $u_1(0)=0$ and that the solution $u_2(x)=-4+3x+x^3$ satisfies $u_2(1)=0$. Take the following ansatz for the Green's function:

$$G(x;\xi)=\begin{cases} c_1u_1(x) & \text{if } 0\le x\le\xi\le 1,\\ c_2u_2(x) & \text{if } 0\le\xi\le x\le 1.\end{cases}$$

Requiring continuity at $x=\xi$ implies that

$$c_2u_2(\xi)-c_1u_1(\xi)=0$$

At $x=\xi$ the derivative $\partial G/\partial x$ must have a jump discontinuity of magnitude $1/p(\xi)=1+\xi^2$, which implies that

$$c_2u_2'(\xi)-c_1u_1'(\xi)=1+\xi^2.$$

Therefore, we obtain the following system of equations:

$$\begin{bmatrix}-u_1(\xi) & u_2(\xi)\\ -u_1'(\xi) & u_2'(\xi)\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}=\begin{bmatrix}0\\ 1+\xi^2\end{bmatrix}.$$

Solving gives

$$\begin{aligned}\begin{bmatrix}c_1\\ c_2\end{bmatrix} &= \frac{1}{u_1'(\xi)u_2(\xi)-u_1(\xi)u_2'(\xi)}\begin{bmatrix}u_2'(\xi) & -u_2(\xi)\\ u_1'(\xi) & -u_1(\xi)\end{bmatrix}\begin{bmatrix}0\\ 1+\xi^2\end{bmatrix}\\ &= \frac{1}{-12(1+\xi^2)}\begin{bmatrix}u_2(\xi)(1+\xi^2)\\ u_1(\xi)(1+\xi^2)\end{bmatrix}\\ &= -\frac{1}{12}\begin{bmatrix}u_2(\xi)\\ u_1(\xi)\end{bmatrix}.\end{aligned}$$

In conclusion, the Green's function is given by

$$G(x;\xi) = -\frac{1}{12}\cdot\begin{cases}(3x+x^3)(-4+3\xi+\xi^3) & \text{if } 0\le x\le \xi\le 1,\\ (3\xi+\xi^3)(-4+3x+x^3) & \text{if } 0\le \xi\le x\le 1.\end{cases}$$

General advice: plug in formulas as late as possible to avoid clutter-induced mistakes.

Solution of problem 7

The Green's function is constructed by setting

$$G(x,y;\xi,\eta) = G_0(x,y;\xi,\eta) + z(x,y;\xi,\eta),$$

where z is harmonic on Ω and satisfies $z = -G_0$ on $\partial\Omega$.

To a point $(\xi,\eta)\in\Omega$ we associate an image point $(\xi',\eta')\in\mathbb{R}^2\setminus\overline{\Omega}$ and set

$$z(x,y;\xi,\eta) = \frac{a}{2\pi}\log\|(x,y)-(\xi',\eta')\| + \frac{b}{2\pi}.$$

This choice guarantees that z is harmonic in Ω. Now we have to determine the constants a and b such that the condition $z=-G_0$ on $\partial\Omega$ is satisfied.

The boundary of Ω is given by $\{(x,x) : x\in\mathbb{R}$. If we define $(\xi',\eta') = (\eta,\xi)$, which is the reflection of (ξ,η) through the line $y=x$, then we have

$$\|(x,x)-(\xi',\eta')\| = \|(x,x)-(\eta,\xi)\| = \|(x,x)-(\xi,\eta)\|$$

for all $x\in\mathbb{R}$. This implies that the condition $z=-G_0$ on $\partial\Omega$ is satisfied when $a=1$ and $b=0$. Therefore, the Green's function is given by

$$\begin{aligned}G(x,y;\xi,\eta) &= -\frac{1}{2\pi}\log\|(x,y)-(\xi,\eta)\| + \frac{1}{2\pi}\log\|(x,y)-(\eta,\xi)\|\\ &= \frac{1}{4\pi}\log\frac{(x-\eta)^2+(y-\xi)^2}{(x-\xi)^2+(y-\eta)^2}.\end{aligned}$$

Solution of problem 8

Taking Fourier transforms gives the following ordinary differential equation:

$$\frac{d^2\widehat{u}}{dt^2} + 2\frac{d\widehat{u}}{dt} + \widehat{u} = 0.$$

The general solution is given by

$$\widehat{u}(t,k) = \widehat{a}(k)e^{-t} + \widehat{b}(k)te^{-t}.$$

From the initial conditions we obtain

$$\widehat{a}(k) = \widehat{f}(k) \quad\text{and}\quad \widehat{b}(k)-\widehat{a}(k) = \widehat{g}(k)$$

and thus

$$\widehat{u}(t,k) = \widehat{f}(k)e^{-t} + (\widehat{f}(k)+\widehat{g}(k))te^{-t}.$$

Taking the inverse Fourier transform gives

$$u(t,x) = f(x)e^{-t} + (f(x)+g(x))te^{-t}.$$